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Averaging of unsteady flows in heterogeneous media of stationary conductivity

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A procedure for deriving equations of average unsteady flows in random media of stationary conductivity is developed. The approach is based on applying perturbation methods in the Fourier–Laplace domain. The main result of the paper is the formulation of an effective Darcy's Law relating the mean velocity to the mean head gradient. In the Fourier–Laplace domain the averaged Darcy's Law is given by a linear local relation. The coefficient of proportionality depends only on the heterogeneity structure and is called the effective conductivity tensor. In the physical domain this relation has a non-local structure and it defines the effective conductivity as an integral operator of convolution type in time and space. The mean head satisfies an unsteady integral-differential equation. The kernel of the integral operator is the inverse Fourier–Laplace transform (FLT) of the effective conductivity tensor. The FLT of the mean head is obtained as a product of two functions: the first describes the FLT of the mean head distribution in a homogeneous medium; the second corrects the solution in a homogeneous medium for the given spatial distribution of heterogeneities. This function is simply related to the effective conductivity tensor and determines the fundamental solution of the governing equation for the mean head. These general results are applied to derive the effective conductivity tensor for small variances of the conductivity. The properties of unsteady average flows in isotropic media are studied by analysing a general structure of the effective Darcy's Law. It is shown that the transverse component of the effective conductivity tensor does not affect the mean flow characteristics. The effective Darcy's Law is obtained as a convolution integral operator whose kernel is the inverse FLT of the effective conductivity longitudinal component. The results of the analysis are illustrated by calculating the effective conductivity for one-, two- and three-dimensional flows. An asymptotic model of the effective Darcy's Law, applicable for distances from the sources of mean flow non-uniformity much larger than the characteristic scale of heterogeneity, is developed. New bounds for the effective conductivity tensor, namely the effective conductivity tensor for steady non-uniform average flow and the arithmetic mean, are proved for weakly heterogeneous media.

1. Introduction

Natural water- and oil-bearing formations exhibit significant erratic spatial variabilities in their ability to conduct fluids. In particular, it is common for the formation conductivity to vary by several orders of magnitude over the space. Such changes in spatial distribution of the conductivity field greatly affect the fluid flow and make the problem much more complicated. Moreover the flow problem constitutes a preliminary step for describing other processes occurring in porous formations. Thus, the spatial distribution of phases and the efficiency of oil displacement depend in many respects on the structure of the heterogeneity. For transport of solute in groundwater, the variations in the fluid velocity control the macro-scale dispersion of the contaminant, phenomenon of great concern in recovering aquifers from pollutant spills (Dagan 1984; Cvetkovic & Dagan 1994).

Flows of fluid in heterogeneous media have been studied intensively during the last three decades as a result of understanding the influence of heterogeneity on the flow and transport properties. The accepted approach to modelling flows in heterogeneous media is the stochastic one, that regards the conductivity as a random space function. In the stochastic approach the solution is represented as a set of all moments of the flow variables. In practice, however, the study is restricted by deriving mathematical models which provide only the few first moments. A central problem is to determine the governing equations for the average flow. For linear constitutive relationship, the problem resides in the averaging of the constitutive equation and determining the effective properties of the process.

This paper addresses the problem of averaging of unsteady flows of water in random porous media of stationary conductivity. Mathematically equivalent problems are encountered in various other physical fields, e.g. heat transfer, electromagnetism and electricity (Batchelor 1974; Beran 1968).

We consider flow in a porous medium whose conductivity $K(\boldsymbol{x})$ is regarded as a random stationary function of spatial coordinate $\boldsymbol{x}$. The simplest case widely reported in the literature is of steady-state flows, uniform in the average. The governing equations are the Darcy's Law $\boldsymbol{v}^{st} = -K\boldsymbol{E}^{st}$ relating the local velocity $\boldsymbol{v}^{st}$ to the local pressure head gradient $\boldsymbol{E}^{st} = \nabla h^{st}$ (h^{st} being the hydraulic head) and the mass conservation balance $\nabla \cdot \boldsymbol{v}^{st} = 0$. The aim is to derive equations satisfied by the mean pressure head $\langle h^{st} \rangle$ and the mean velocity $\langle \boldsymbol{v}^{st} \rangle$. Since the mass balance equation is easily averaged to yield $\nabla \cdot \langle \boldsymbol{v}^{st} \rangle = 0$, the problem is the averaging of the Darcy's Law, i.e. deriving a relation between the mean pressure head gradient and the mean velocity. It is emphasized that only relations which are independent of the boundary conditions are sought. From a mathematical point of view this is possible only for infinite flow domains. In practice, they are applicable in the flow subdomains far from the sources of flow non-uniformity. Thus, for weakly heterogeneous media and for mean uniform flow in a bounded domain, the averaged Darcy's Law is valid at distances of a few scales of the log-conductivity heterogeneities from the boundaries (Rubin & Dagan 1988, 1989). We neglect the effects of boundaries and consider the flow in infinite domains. Then, the assumption of the mean flow uniformity is satisfied if the boundary condition is $\lim_{x \to \infty} h^{st}(\boldsymbol{x}) = \langle \boldsymbol{E}^{st} \rangle \cdot \boldsymbol{x}$ where $\langle \boldsymbol{E}^{st} \rangle$ is a constant mean head gradient. In this case the averaged Darcy's Law has the form

$$\langle v_m^{st} \rangle = -K_{ml}^{eff} \langle E_l^{st} \rangle, \tag{1}$$

where the constant tensor $\boldsymbol{K}^{eff}$ depends only on the statistics of $\boldsymbol{K}(\boldsymbol{x})$ and is called the tensor of effective conductivity. The following best bounds are known for $\boldsymbol{K}^{eff}$ (e.g. Batchelor 1974; Matheron 1967*b*):

$$K_H \boldsymbol{I} \leqslant \boldsymbol{K}^{eff} \leqslant K_A \boldsymbol{I}, \tag{2}$$

where K_H and K_A are the harmonic and arithmetic means, respectively, and $\boldsymbol{I}$ is the unit matrix. It is emphasized that for given structure of heterogeneity, $\boldsymbol{K}^{eff}$ is a constant in the flow domain. The bounds (2) are attained in stratified porous media for flows parallel and normal to the direction of layering. Considerable efforts have been undertaken to calculate the effective conductivity tensor for different structures of media. Except for a few exact closed formulae, all results were obtained under various assumptions regarding the fluctuations of the conductivity field and they are reported in a few reviews and articles (e.g. Beran 1968; Dagan 1989; Shvidler 1985).

In most applications, e.g. in the presence of space-distributed sources, the assumption of flow uniformity fails and relation (1) is applicable only to the flow subdomains far from the wells. Moreover, for elastic media and for time-dependent boundary conditions, the flow is no longer steady. The main question is whether there exists a relation between the mean velocity and the mean head gradient which, similarly to the uniform steady flow, depends only on the statistics of K. Although of a fundamental nature, this problem has received little attention in the literature.

Several investigators have studied flow toward a single source of given deterministic strength. The solution of the equation for the pressure head is given by the Green function of the corresponding operator. The Green function is random, since such is the conductivity K. The averaged Green function describes the space distribution of the mean head for the flow toward a source and is referred to as a mean Green function. Shvidler (1966, 1985) has developed a perturbation analysis for steady flows in weakly heterogeneous media and has derived two asymptotical limits of the mean Green function at distances from the singular point much smaller and greater than the heterogeneity scale. He has shown that far from the well the medium can be treated as a homogeneous one of conductivity equal to the effective conductivity for steady uniform flow. In contrast, the mean head distribution in the vicinity of the well is described by the solution for a homogeneous medium with the conductivity equal to K_H. The same results were obtained for unsteady flows at large times (Shvidler 1966). Shvidler's approach was expanded by Matheron (1967*a*, *b*) who derived the ratio between a mean specific discharge and a mean head gradient in a one-well-driven system. Matheron showed that for non-uniform steady flows a constant effective conductivity depending only on the statistics of K could not be defined. Later Freeze (1975) arrived to the same conclusion for one-dimensional unsteady flow. Several attempts (Ababou & Wood 1990; Naff 1991) have been made to derive expressions for the effective conductivity applicable for large variance of the logconductivity by regarding the Shvidler–Matheron results for steady flows as a first-order exponential series expansion. However, thorough numerical simulations by the Monte Carlo method (Desbarats 1992; Neuman & Orr 1993) did not confirm the conclusions of Ababou & Wood (1990) and were in contradiction with the approach of Naff (1991).

It is emphasized that the effective conductivity determined in the above-mentioned investigations is obtained as a function of the coordinate vector and it is not applicable to distributions of sources other than the one-well system. In other words the relationship between the mean velocity and head gradient does not represent a constitutive equation. For steady-state flows a dependence of the mean flux on the mean head gradient was suggested in the general form of a linear non-local functional by Saffman (1971). Following the same reasoning Dagan (1989) discussed the expression for the mean flux on assuming that the mean head gradient is continuous and varies slowly. The first-order expression for the mean velocity in the logconductivity variance was derived by expanding the mean head gradient in a Taylor series. The averaging of unsteady flows without sources has been carried out for a case of small variations in the initial space distribution of the head gradient (Dagan 1982). The effective Darcy's Law was obtained in a local form with effective conductivity changing from the arithmetic mean to the effective uniform value with growing time. The investigation of the averaged non-uniform steady-state flows in bounded domains was conducted by Neuman & Orr (1993). They show that the relationship between the mean velocity and the mean head gradient is non-local.

The main objective of the present paper is to derive a general mathematical model of the average unsteady flows and to study their properties. This is achieved by a perturbation expansion in the Fourier–Laplace domain and subsequent solution of the equations for perturbations. The effective Darcy's Law is obtained by summing the perturbation series and leads to the definition of the effective conductivity. These general results are studied further for weakly heterogeneous media by deriving the first-order approximation of the effective Darcy's Law in the conductivity variance.

2. Mathematical statement of the problem

We consider the unsteady flow of a homogeneous fluid in a heterogeneous unbounded domain. The Darcy's velocity $\boldsymbol{v}$ and the hydraulic head h obey the following system of equations:

$$S\frac{\partial h(\boldsymbol{x},\bar{t})}{\partial \bar{t}}+\nabla\cdot \boldsymbol{v}(\boldsymbol{x},\bar{t}) = \bar{\phi}(\boldsymbol{x},\bar{t}), \tag{3}$$

$$\boldsymbol{v}(\boldsymbol{x},\bar{t}) = -K(\boldsymbol{x})\,\boldsymbol{E}(\boldsymbol{x},\bar{t});\quad \boldsymbol{E}(\boldsymbol{x},\bar{t}) = \nabla h(\boldsymbol{x},\bar{t}), \tag{4}$$

where K is the hydraulic conductivity and $S = s/n$ with s and n being the specific storativity and the porosity. A simple example of the source term $\bar{\phi}$ in (3) is a function of finite support in space, e.g. a system of singular sources and sinks. To simplify matters the storativity and the porosity are assumed deterministic and constant with n absorbed in the definition of $\bar{\phi}$. The conductivity K is regarded as a stationary random space function of given statistics.

We introduce a random space function $\epsilon(\boldsymbol{x})$ as

$$K(\boldsymbol{x}) = K_A[1-\epsilon(\boldsymbol{x})];\quad \langle\epsilon\rangle = 0, \tag{5}$$

where $K_A = \langle K\rangle$ is the mean conductivity. It is seen that the variance σ^2 of ϵ is equal to the coefficient of variation of K, whereas the $(n+1)$-point correlation functions of ϵ

$$\rho(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_n) = \langle\epsilon(\boldsymbol{x})\,\epsilon(\boldsymbol{x}+\boldsymbol{r}_1)\ldots\epsilon(\boldsymbol{x}+\boldsymbol{r}_n)\rangle\quad (n = 1,2,\ldots) \tag{6}$$

are simply expressed through the $(n+1)$-point correlation functions of K. In particular, the autocorrelation function $\rho(\boldsymbol{r})$ is identical to that of K.

Defining $t = K_A\,\bar{t}/S$, the system (3) and (4) leads to the following equation for h:

$$\frac{\partial h(\boldsymbol{x},t)}{\partial t}-\nabla^2 h(\boldsymbol{x},t) = -\nabla\cdot[\epsilon(\boldsymbol{x})\,\nabla h(\boldsymbol{x},t)]+\phi(\boldsymbol{x},t), \tag{7}$$

where $\phi = \bar{\phi}/K_A$.

The solution of (7) is sought subject to the arbitrary deterministic initial condition

$$h(\boldsymbol{x},t)\,|_{t=0} = h_0(\boldsymbol{x}), \tag{8}$$

and to the boundary condition $\lim_{x\to\infty} h(\boldsymbol{x}) = 0$ where $x = |\boldsymbol{x}|$.

Equation (7) is solved by applying the Laplace transform (LT) in time and the Fourier transform (FT) in space where the latter is defined by

$$\hat{\epsilon}(\boldsymbol{k}) = \int d\boldsymbol{x}\,\epsilon(\boldsymbol{x})\exp(i\boldsymbol{k}\cdot\boldsymbol{x}).$$

The following notation is adopted in the paper: in the Fourier domain we use the independent variables $\boldsymbol{k}$ or $\boldsymbol{p}$, whereas the variable of the LT is λ. The FT of the function $f(\boldsymbol{x}, t)$ is denoted by a tilded quantity $\tilde{f}(\boldsymbol{k}, t)$. The LT is referred to as a 'checked' function $\check{f}(\boldsymbol{x}, \lambda)$. Applying the Fourier–Laplace transform (FLT) leads to the 'hatted' function $\hat{f}(\boldsymbol{k}, \lambda)$ of $\boldsymbol{k}$ and λ.

The FLT of the head satisfies the integral equation resulting from (7) and (8)

$$\hat{h}(\boldsymbol{k}, \lambda) = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^d} \frac{\boldsymbol{k} \cdot \boldsymbol{p}}{k^2 + \lambda} \tilde{\epsilon}(\boldsymbol{k} - \boldsymbol{p}) \hat{h}(\boldsymbol{p}, \lambda) + \frac{\hat{\phi}(\boldsymbol{k}, \lambda) + \tilde{h}_0(\boldsymbol{k})}{k^2 + \lambda}, \tag{9}$$

where $\hat{\phi}(\boldsymbol{k}, \lambda)$ is the FLT of ϕ, $\tilde{\epsilon}(\boldsymbol{k})$ and $\tilde{h}_0(\boldsymbol{k})$ are the FT of $\epsilon(\boldsymbol{x})$ and $h_0(\boldsymbol{x})$, respectively, and d is the space dimensionality.

The solution of (9) is sought by expanding the head's FLT in a powers series in ϵ:

$$\hat{h}(\boldsymbol{k}, \lambda) = \hat{h}^{(0)}(\boldsymbol{k}, \lambda) + \hat{h}^{(1)}(\boldsymbol{k}, \lambda) + \hat{h}^{(2)}(\boldsymbol{k}, \lambda) + \dots; \quad \hat{h}^{(m)}(\boldsymbol{k}, \lambda) = O(\tilde{\epsilon}^m). \tag{10}$$

Substituting (10) in (9) and collecting terms of the same order leads to the following system of equations:

$$\hat{h}^{(n+1)}(\boldsymbol{k}, \lambda) = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^d} \frac{\boldsymbol{k} \cdot \boldsymbol{p}}{k^2 + \lambda} \tilde{\epsilon}(\boldsymbol{k} - \boldsymbol{p}) \hat{h}^{(n)}(\boldsymbol{p}, \lambda); \quad n = 0, 1, \dots, \tag{11}$$

$$\hat{h}^{(0)}(\boldsymbol{k}, \lambda) = \frac{\hat{\phi}(\boldsymbol{k}, \lambda) + \tilde{h}_0(\boldsymbol{k})}{k^2 + \lambda}. \tag{12}$$

The system (10)–(12) determines the FLT of the head. Note that the inverse FLT of the zero-order approximation $\hat{h}^{(0)}(\boldsymbol{k}, \lambda)$ yields the head distribution in a homogeneous media of constant conductivity K_A. The system (11) adjusts the 'homogeneous' solution (12) to the actual one with increasing order of accuracy in the conductivity fluctuations with the $(n+1)$-order correction given by the convolution integral from the n-order one. As we shall show in the next section this property allows one to derive the effective Darcy's Law for average unsteady flow in media of stationary conductivity.

The steady-state distribution of the head is determined by applying the inverse FT to the function $\tilde{h}^{st}(\boldsymbol{k}) = \lim_{t \to \infty} \tilde{h}(\boldsymbol{k}, t) = \lim_{\lambda \to 0} \lambda \hat{h}(\boldsymbol{k}, \lambda)$. It is seen that the FT of $h^{st}(\boldsymbol{x})$ is obtained from (10)–(12) by setting $\lambda = 0$ and $\tilde{h}_0(\boldsymbol{k}) = 0$.

3. Average flow equations

We proceed now with deriving the averaged Darcy's Law for unsteady flow in media of stationary conductivity. At the same time we determine the general solution of the average flow problem, namely the distribution of the mean head in space and time. This is carried out by solving (11) to consecutive orders for $n = 0, 1, \dots$ and by substituting the results in (10). The expected value of the latter yields the expression for the FLT of the mean head $\langle h \rangle$. The relation between the FLTs of the mean velocity and the mean head gradient is derived in a similar way. The details of the derivation are given in Appendix A. Herein we summarize the main results of Appendix A.

Thus, the effect of heterogeneity on the average flow is manifested through the tensorial function $\boldsymbol{\Omega}(\boldsymbol{k}, \lambda)$ defined by a series

$$\boldsymbol{\Omega}(\boldsymbol{k}, \lambda) = \sum_{n=2}^{\infty} \boldsymbol{\Omega}^{(n)}(\boldsymbol{k}, \lambda); \quad \boldsymbol{\Omega}^{(n)} = O(\sigma^n). \tag{13}$$

The second-order tensors $\boldsymbol{\Omega}^{(n+1)}$ in series (13) depend on the FT of the $(n+1)$-point correlation function (6):

$$\Phi(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_{n+1}) = \int\cdots\int \mathrm{d}\boldsymbol{r}_1\ldots\mathrm{d}\boldsymbol{r}_n\,\rho(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_n)\exp\left[\mathrm{i}\sum_{m=1}^{n}\boldsymbol{r}_m\cdot(\boldsymbol{k}_m-\boldsymbol{k}_{m+1})\right], \tag{14}$$

and their components are given by

$$\Omega_{ml}^{(n+1)}(\boldsymbol{k},\lambda) = \int\cdots\int\frac{\mathrm{d}\boldsymbol{p}_1\ldots\mathrm{d}\boldsymbol{p}_n}{(2\pi)^{nd}}\frac{p_{1,m}p_{n,l}}{p_n^2+\lambda}\frac{\boldsymbol{p}_1\cdot\boldsymbol{p}_2}{p_1^2+\lambda}\cdots\frac{\boldsymbol{p}_{n-1}\cdot\boldsymbol{p}_n}{p_{n-1}^2+\lambda}\Phi(\boldsymbol{k},\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n) \tag{15}$$

$$(n = 1,2,\ldots;\quad m,l = 1,\ldots,d).$$

The FLT of the mean head is obtained in Appendix A as

$$\langle\hat{h}(\boldsymbol{k},\lambda)\rangle = \frac{\hat{h}^{(0)}(\boldsymbol{k},\lambda)}{\hat{Y}(\boldsymbol{k},\lambda)}, \tag{16}$$

where $\hat{h}^{(0)}$ is given by (12) and the function $\hat{Y}$ is defined as the normalized contraction of Ω with respect to the wave vector $\boldsymbol{k}$:

$$\hat{Y}(\boldsymbol{k},\lambda) = \left[1+\frac{\boldsymbol{k}\cdot\boldsymbol{\Omega}(\boldsymbol{k},\lambda)\cdot\boldsymbol{k}}{k^2+\lambda}\right]^{-1}. \tag{17}$$

Owing to the nature of $\boldsymbol{\Omega}$ the function $\hat{Y}$ depends only on the heterogeneity structure.

The solution (16) determines the FLT of the mean head. It shows how the initial condition and the sources are represented by the term $\hat{h}^{(0)}$, whereas the structure of heterogeneity is embedded in the function $\hat{Y}$. For homogeneous media $\hat{Y}=1$ and (16) yields the solution of (3), (4) with $K(x) = K_A$.

The effective Darcy's Law in the Fourier–Laplace space is given by a linear relation between the FLTs of the mean velocity and head gradient:

$$\langle\hat{v}_m(\boldsymbol{k},\lambda)\rangle = -\hat{K}_{ml}^{eff}(\boldsymbol{k},\lambda)\langle\hat{E}_l(\boldsymbol{k},\lambda)\rangle, \tag{18}$$

where

$$\hat{K}_{ml}^{eff}(\boldsymbol{k},\lambda) = K_A[\delta_{ml}-\hat{Y}(\boldsymbol{k},\lambda)\,\Omega_{ml}(\boldsymbol{k},\lambda)]. \tag{19}$$

It follows from (13)–(15), (17) and (19) that $\hat{\boldsymbol{K}}^{eff}(\boldsymbol{k},\lambda)$ depends only on the statistics of the conductivity K. It is easy to check using (17) and (19) that the function $\hat{Y}(\boldsymbol{k},\lambda)$ is related to normalized contraction of the tensor $\hat{\boldsymbol{K}}^{eff}(\boldsymbol{k},\lambda)$ as follows:

$$\hat{Y}(\boldsymbol{k},\lambda) = \frac{\boldsymbol{k}\cdot\hat{\boldsymbol{K}}^{eff}(\boldsymbol{k},\lambda)\cdot\boldsymbol{k}}{K_A(k^2+\lambda)}+\frac{\lambda}{k^2+\lambda}. \tag{20}$$

Expressions (13)–(19) represent a general solution of the problem of average unsteady flow in the Fourier–Laplace space. Given the initial conditions and source function, the mean head distribution is obtained by applying the inverse FLT to (16). The fundamental property of this solution is a local dependence of the velocity's FLT on the FLT of the mean head gradient. Applying the inverse FLT to (18) yields the relation between the mean velocity and the mean head gradient in the form of a temporal–spatial convolution integral:

$$\langle v_m(\boldsymbol{x},t)\rangle = -\int_0^t \mathrm{d}\tau\int \mathrm{d}\boldsymbol{x}'\,K_{ml}^{eff}(\boldsymbol{x}-\boldsymbol{x}',t-\tau)\langle E_l(\boldsymbol{x}',\tau)\rangle. \tag{21}$$

In (21) the kernel $K_{ml}^{eff}(\boldsymbol{x},t)$ is the inverse FLT of $\hat{K}_{ml}^{eff}(\boldsymbol{k},\lambda)$ and therefore is completely determined by the statistics of $K(x)$. Since (21) is applicable for any deterministic source function $\phi(\boldsymbol{x}, t)$ and any initial condition $h_0(\boldsymbol{x})$, relation (21) defines the effective conductivity as a convolution integral operator in space and time. The kernel of this operator is a tensorial function of $\boldsymbol{x}$ and t whose FLT is given by (19). The function $\hat{\boldsymbol{K}}^{eff}(\boldsymbol{k}, \lambda)$ (19) is referred to as effective conductivity tensor.

It is important to note that (21) was obtained for a potential vector $\langle \boldsymbol{E} \rangle$

$$\langle \boldsymbol{E}(\boldsymbol{x}, t)\rangle = \nabla \langle \boldsymbol{h}(\boldsymbol{x}, t)\rangle. \tag{22}$$

The condition (22) does not render a unique $\boldsymbol{K}^{eff}(\mathbf{x}, t)$ and $\hat{\boldsymbol{K}}^{eff}(\boldsymbol{k}, \lambda)$. Thus, any tensor $\boldsymbol{A}(\boldsymbol{x}, t)$ satisfying a divergent-free condition $\nabla \cdot \boldsymbol{A}(\boldsymbol{x}, t) = 0$ (or its equivalent $\hat{A}_{ml}(\boldsymbol{k}, \lambda)\, k_l = 0$ in the Fourier–Laplace domain) can be added to the kernel $\boldsymbol{K}^{eff}(\boldsymbol{x}, t)$ (the effective conductivity tensor $\hat{\boldsymbol{K}}^{eff}(\boldsymbol{k}, \lambda)$, respectively) such that the effective Darcy's Law is not affected. Indeed,

$$\begin{aligned}
-\int_0^t \mathrm{d}\tau \int \mathrm{d}\boldsymbol{x}' \, [K_{ml}^{eff}(\boldsymbol{x}-\boldsymbol{x}', t-\tau) + A_{ml}(\boldsymbol{x}-\boldsymbol{x}', t-\tau)] \frac{\partial \langle h(\boldsymbol{x}', \tau)\rangle}{\partial x_l'} \\
= -\int_0^t \mathrm{d}\tau \int \mathrm{d}\boldsymbol{x}' \, K_{ml}^{eff}(\boldsymbol{x}-\boldsymbol{x}', t-\tau) \frac{\partial \langle h(\boldsymbol{x}', \tau)\rangle}{\partial x_l'} \\
+ \int_0^t \mathrm{d}t' \int \mathrm{d}\boldsymbol{x}' \frac{\partial A_{ml}(\boldsymbol{x}-\boldsymbol{x}', t-\tau)}{\partial x_l'} \langle h(\boldsymbol{x}', \tau)\rangle \\
= -\int_0^t \mathrm{d}\tau \int \mathrm{d}\boldsymbol{x}' \, K_{ml}^{eff}(\boldsymbol{x}-\boldsymbol{x}', t-\tau) \langle E_l(\boldsymbol{x}', \tau)\rangle = \langle v_l(\boldsymbol{x}, t)\rangle.
\end{aligned}$$

It is easy to see from (20) that this non-uniqueness of the effective conductivity tensor does not affect the function $\hat{Y}$ and the mean head distribution.

The effective Darcy's Law (21) coupled with the averaged mass balance equation

$$\frac{\partial \langle h(\boldsymbol{x}, t)\rangle}{\partial t} + \nabla \cdot \langle \boldsymbol{v}(\boldsymbol{x}, t)\rangle = \phi(\boldsymbol{x}, t) \tag{23}$$

constitute a mathematical model of average unsteady flows in stationary media. The mean head obeys the linear integro-differential equation resulting from substitution of (21) and (22) into (23):

$$\frac{\partial \langle h(\boldsymbol{x}, t)\rangle}{\partial t} - \frac{\partial}{\partial x_m} \int_0^t \mathrm{d}\tau \int \mathrm{d}\boldsymbol{x}' \, K_{ml}^{eff}(\boldsymbol{x}-\boldsymbol{x}', t-\tau) \frac{\partial \langle h(\boldsymbol{x}', \tau)\rangle}{\partial x_l'} = \phi(\boldsymbol{x}, t), \tag{24}$$

and is determined uniquely for given initial coindition (8) and source function ϕ. The fundamental solution of (24) corresponds to the flow toward a single source and constant initial head distribution, i.e. $\phi(\boldsymbol{x}, t) = \delta(\boldsymbol{x})\,\delta(t)$ and $h_0(\boldsymbol{x}) = 0$. This solution $G(\boldsymbol{x}, t)$ is called the mean Green function. Its FLT $\hat{G}(\boldsymbol{k}, \lambda)$ is completely determined by the contraction $\hat{Y}(\boldsymbol{k}, \lambda)$ and results from (12) and (16) as follows:

$$\hat{G}(\boldsymbol{k}, \lambda) = \frac{1}{\hat{Y}(\boldsymbol{k}, \lambda)\,(k^2 + \lambda)}. \tag{25}$$

For the purpose of illustration, let consider the important case of a flow toward a single pumping well of discharge $Q(t)$. The FLT of the mean head is given by

$$\langle \hat{h}(\boldsymbol{k}, \lambda)\rangle = \check{Q}(\lambda)\, \hat{G}(\boldsymbol{k}, \lambda), \tag{26}$$

where $\check{Q}(\lambda)$ is the LT of $Q(t)$. The inverse FLT of (26) yields the distribution of the mean head in space and time.

4. Mathematical model of average steady flow

The mathematical model of steady-state flows is obtained from the results of the previous section by setting $\lambda = 0$. In the Fourier space it yields

$$\langle \tilde{h}^{st}(\boldsymbol{k}) \rangle = \frac{\tilde{\phi}(\boldsymbol{k})}{\tilde{Y}(\boldsymbol{k})\, k^2}, \tag{27}$$

$$\langle \tilde{v}^{st}_m(\boldsymbol{k}) \rangle = -\tilde{K}^{eff}_{ml}(\boldsymbol{k}) \langle \tilde{E}^{st}_l(\boldsymbol{k}) \rangle, \tag{28}$$

$$\tilde{K}^{eff}_{ml}(\boldsymbol{k}) = K_A [\delta_{ml} - \tilde{Y}(\boldsymbol{k})\, \Omega_{ml}(\boldsymbol{k})], \tag{29}$$

$$\tilde{Y}(\boldsymbol{k}) = [1 + \boldsymbol{k} \cdot \boldsymbol{\Omega}(\boldsymbol{k}) \cdot \boldsymbol{k}]^{-1}, \tag{30}$$

where $\boldsymbol{\Omega}(\boldsymbol{k}) = \boldsymbol{\Omega}(\boldsymbol{k}, \lambda = 0)$. The expression for the effective conductivity tensor (20) is simplified to

$$\tilde{Y}(\boldsymbol{k}) = \frac{\boldsymbol{k} \cdot \tilde{\boldsymbol{K}}^{eff}(\boldsymbol{k}) \cdot \boldsymbol{k}}{K_A k^2}. \tag{31}$$

In the physical domain the effective Darcy's Law is obtained from (28) in the form of the spatial convolution integral

$$\langle v^{st}_m(\boldsymbol{x}) \rangle = -\int \mathrm{d}\boldsymbol{x}'\, K^{eff}_{ml}(\boldsymbol{x} - \boldsymbol{x}') \langle E^{st}_l(\boldsymbol{x}') \rangle, \tag{32}$$

where the kernel $\boldsymbol{K}^{eff}(\boldsymbol{x})$ is the inverse FT of $\tilde{\boldsymbol{K}}^{eff}(\boldsymbol{k})$, (29). The mean velocity $\langle \boldsymbol{v} \rangle$ obeys the equation $\boldsymbol{\nabla} \cdot \langle \boldsymbol{v}(\boldsymbol{x}) \rangle = \phi(\boldsymbol{x})$. The steady mean Green function results from (27) as follows:

$$\tilde{G}^{st}(\boldsymbol{k}) = \frac{1}{\tilde{Y}(\boldsymbol{k})\, k^2}. \tag{33}$$

The results of this section have been derived previously by direct averaging the equations of steady flow (Indelman & Abramovich 1994*b*).

5. General analysis of the averaged Darcy's Law

The objective of this section is to study general properties of the effective Darcy's Law. Starting with the steady case, the effective conductivity defined by (29) generalizes the concept of effective property for uniform average flows. Indeed, for the uniform flow case $\langle \boldsymbol{E} \rangle = \langle \boldsymbol{\nabla} h \rangle = \text{const}$ and (32) lead to the local averaged Darcy's Law

$$\langle v^{st}_m \rangle = \int \mathrm{d}\boldsymbol{x}\, K^{eff}_{ml}(\boldsymbol{x}) \langle E^{st}_l \rangle = \tilde{K}^{eff}_{ml}(\boldsymbol{k} = 0) \langle E^{st}_l \rangle. \tag{34}$$

Comparison of (34) and (1) yields

$$\boldsymbol{K}^{eff} = \tilde{\boldsymbol{K}}^{eff}(0), \tag{35}$$

i.e. the effective conductivity for mean uniform steady flow is the value of the effective conductivity tensor $\tilde{\boldsymbol{K}}^{eff}(\boldsymbol{k})$ of non-uniform steady flow at $\boldsymbol{k} = 0$.

Similarly, (32) may be obtained from (21). In Fourier space we have

$$\langle \tilde{v}^{st}_m(\boldsymbol{k}) \rangle = \lim_{\lambda \to 0} \lambda \langle \hat{v}_m(\boldsymbol{k}, \lambda) \rangle = -\hat{K}^{eff}_{ml}(\boldsymbol{k}, \lambda = 0) \lim_{\lambda \to 0} \lambda \langle \hat{E}_l(\boldsymbol{k}, \lambda) \rangle = -\tilde{K}^{eff}_{ml}(\boldsymbol{k}) \langle \tilde{E}^{st}_l(\boldsymbol{k}) \rangle. \tag{36}$$

It is seen from (35) that the uniform effective conductivity $\boldsymbol{K}^{eff}$ is the value of the ECT of unsteady flow at $\boldsymbol{k} = 0$ and $\lambda = 0$. Thus at large times

$$\tilde{\boldsymbol{K}}^{eff}(\boldsymbol{k}) = \hat{\boldsymbol{K}}^{eff}(\boldsymbol{k}, 0); \quad \boldsymbol{K}^{eff} = \tilde{\boldsymbol{K}}^{eff}(0) = \hat{\boldsymbol{K}}^{eff}(0, 0). \tag{37}$$

In contrast, for small times the effective Darcy's Law takes the form

$$\lim_{t \to 0} \langle \tilde{v}_m(\boldsymbol{k}, t) \rangle = -\hat{K}^{eff}_{ml}(\boldsymbol{k}, \lambda = \infty) \lim_{t \to 0} \langle \hat{E}_l(\boldsymbol{k}, t) \rangle = -K_A \lim_{t \to 0} \langle \hat{E}_l(\boldsymbol{k}, t) \rangle, \tag{38}$$

i.e. the effective conductivity of the medium is equal to the arithmetic mean.

Excluding several special cases of the conductivity spatial distribution (Neuman & Orr 1993), uniform mean steady-state flow in an unbounded domain is the only case for which the averaged Darcy's Law has a local structure. For any other case relation (21) shows that the mean velocity depends on the distribution of the mean head gradient in the whole flow domain and on the process history.

Note also that in general the effective conductivity tensor depends on the shape of the conductivity correlations. However, there are several cases when this dependence is no longer present and the effective conductivity tensor becomes a function of the conductivity statistics rather than a functional. Thus, we show in Appendix B that for one-dimensional unsteady flow the effective conductivity tensor is a functional of the conductivity multipoint correlations. This dependence on the conductivity statistics vanishes with $t \to \infty$ $(\lambda \to 0)$ and the perturbation series is summed to yield the well-known steady-state effective value K_H. The second case when the dependence on the correlation structure vanishes is a two-dimensional steady uniform flow in an isotropic medium. This case corresponds to the limit $\boldsymbol{k} \to 0$ and $\lambda \to 0$ and for an even p.d.f. of the logconductivity the effective conductivity is equal to the geometric mean (Matheron 1967*b*; Dykhne 1970; Abramovich 1977). The third case corresponds to the initial stage of unsteady flows for which $\lim_{\lambda \to \infty} \hat{\boldsymbol{K}}^{eff}(\boldsymbol{k}, \lambda) = K_A$. These cases exhaust our knowledge about closed formulae for the effective conductivity of continuous random media. It was shown recently (Indelman & Abramovich 1994*a*; Abramovich & Indelman 1995) that the effective property for uniform steady-state flow in a three-dimensional log-normal isotropic media and in anisotropic media of any dimensionality is a functional of the correlation structure. We conclude this discussion by saying that the ECT of unsteady processes is a function of two variables and a functional of the conductivity correlation structure.

Dagan (1982) has studied unsteady flows without sources. Assuming a slowly varying mean head gradient he has arrived at a local averaged Darcy's Law in the form

$$\langle v_m(\boldsymbol{x}, t) \rangle = -K^*_{ml}(t) \langle E_l(\boldsymbol{x}, t) \rangle. \tag{39}$$

The assumption of small variabilities of the mean head gradient in time and space is equivalent to moving $\langle \boldsymbol{E} \rangle$ out of the integrals in (21). This leads to the following expression for K^*_{ml} in (39):

$$K^*_{ml}(t) = \int_0^t \mathrm{d}\tau \, \tilde{K}^{eff}_{ml}(\boldsymbol{k} = 0, \tau). \tag{40}$$

It follows from (36) and (38) that for small times $\lim_{t \to 0} K^*_{ml}(t) = K_A \, \delta_{ml}$ whereas for large times $\lim_{t \to \infty} K^*_{ml}(t) = K^{eff}_{ml}$ in agreement with the first-order results obtained by Dagan (1982).

Summarizing the results obtained so far, a mathematical model of unsteady average flows in random media of stationary conductivity was developed. We have derived a general solution of the governing equations and have shown that the flow parameters of interest (mean head, its gradient, mean velocity) are expressed with the aid of the $d \times d$ tensorial function $\boldsymbol{\Omega}(\boldsymbol{k}, \lambda)$, where $\boldsymbol{k}$ and λ are the Fourier and Laplace variables, respectively. This function depends only on the conductivity statistics and does not depend on the distributions of the sources and the initial head. It determines all the features of the average flows stemming from the medium random heterogeneity. We have shown that $\boldsymbol{\Omega}(\boldsymbol{k}, \lambda)$ is simply related to the effective conductivity tensor and to the fundamental solution of the averaged equation for the mean head which we have called the mean Green function. Although the function $\boldsymbol{\Omega}$ is defined in an explicit manner by the multidimensional integrals over the multivariate conductivity correlators, its rigorous calculation represents a very cumbersome and complicated problem. In the remainder of the paper we apply the general results to weakly heterogeneous media and study the properties of unsteady average flows by deriving a first-order approximation of the effective Darcy's Law in the conductivity variance.

6. First-order analysis

The first-order approximation of the effective conductivity tensor in the variance σ^2 of the fluctuation $\epsilon(\boldsymbol{x})$ results from retaining terms in (19) up to σ^2-order. Since $\boldsymbol{\Omega}$ in (13) is a sum of $\boldsymbol{\Omega}^{(n)}$ and $\boldsymbol{\Omega}^{(n)} = O(\sigma^n)$ the general expression for the effective conductivity tensor (19) simplifies to

$$\hat{K}^{eff}_{ml}(\boldsymbol{k}, \lambda) = K_A[\delta_{ml} - \sigma^2 \hat{s}_{ml}(\boldsymbol{k}, \lambda)], \tag{41}$$

where

$$\hat{s}_{ml}(\boldsymbol{k}, \lambda) = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^d} \frac{p_m p_l}{p^2 + \lambda} \Phi(\boldsymbol{k} - \boldsymbol{p}). \tag{42}$$

Here, $\Phi(\boldsymbol{k})$ is the spectrum of heterogeneity, i.e. the FT of the autocorrelation function $\rho(\boldsymbol{r})$ of K.

The function $\hat{Y}(\boldsymbol{k}, \lambda)$ determining the mean Green function (25) results from (20) as follows:

$$\hat{Y}(\boldsymbol{k}, \lambda) = 1 - \sigma^2 \hat{g}(\boldsymbol{k}, \lambda) \frac{k^2}{k^2 + \lambda}, \tag{43}$$

where $\hat{g}$ is the normalized contraction of the tensor $\hat{\boldsymbol{s}}$ with the vector $\boldsymbol{k}$ and is given by

$$\hat{g}(\boldsymbol{k}, \lambda) = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^d} \frac{(\boldsymbol{k} \cdot \boldsymbol{p})^2}{k^2 (p^2 + \lambda)} \Phi(\boldsymbol{k} - \boldsymbol{p}). \tag{44}$$

The FLT of the mean Green function is simply expressed in terms of the function $\hat{g}$:

$$\hat{G}(\boldsymbol{k}, \lambda) = \frac{1}{k^2 + \lambda} + \sigma^2 \hat{g}(\boldsymbol{k}, \lambda) \frac{k^2}{(k^2 + \lambda)^2}. \tag{45}$$

The first term in (45) is the FLT of the Green function for flow in a homogeneous medium. The second term represents the first-order correction to the flow due to the medium heterogeneity.

Equations (41) and (42) show explicitly the dependence of the effective conductivity tensor on the shape of the correlation. Consider the trace of the effective conductivity tensor resulting from (41):

$$\operatorname{Tr} \hat{\boldsymbol{K}}^{eff}(\boldsymbol{k}, \lambda) = K_A[d - \sigma^2 \hat{\psi}(\boldsymbol{k}, \lambda)], \tag{46}$$

where $\hat{\psi}$ is the trace of the tensor $\hat{\boldsymbol{s}}$:

$$\hat{\psi}(\boldsymbol{k}, \lambda) = \operatorname{Tr} \hat{\boldsymbol{s}}(\boldsymbol{k}, \lambda) = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^d} \frac{p^2}{p^2 + \lambda} \Phi(\boldsymbol{k} - \boldsymbol{p}). \tag{47}$$

It is seen that the trace of the effective conductivity tensor depends on the shape of the correlation as well since the trace of $\hat{\boldsymbol{s}}$ (47) does. Only for small and large times (large and small λ, respectively) does the dependence on the correlation structure vanish and we have

$$\operatorname{Tr}\hat{\boldsymbol{K}}^{eff}(\boldsymbol{k},0) = K_A(d-\sigma^2); \quad \operatorname{Tr}\hat{\boldsymbol{K}}^{eff}(\boldsymbol{k},\infty) = dK_A. \tag{48}$$

The first result in (48) recovers the property of the effective conductivity tensor trace for steady-state flow (Indelman & Abramovich 1994*b*), whereas the second one is in agreement with (38). A second scalar function of interest is the contraction of $\hat{\boldsymbol{K}}^{eff}$ with the vector $\boldsymbol{k}$:

$$\frac{\boldsymbol{k}\cdot\hat{\boldsymbol{K}}^{eff}(\boldsymbol{k},\lambda)\cdot\boldsymbol{k}}{k^2} = K_A[1-\sigma^2\hat{g}(\boldsymbol{k},\lambda)]. \tag{49}$$

It is seen that the normalized contraction of the effective conductivity tensor depends on the correlation function for any finite λ, i.e. for $t>0$.

The results of this Section are applicable to any structure of the conductivity field and to any number of space dimensions. To determine the effective conductivity tensor one has to specify the spectrum Φ and to calculate the tensor $\hat{\boldsymbol{s}}$, (42). As a rule the latter cannot be done analytically and hence numerical integration is required. In the following we apply the results of this section to one-, two- and three-dimensional flows in isotropic media.

7. Effective conductivity of isotropic media

The structure of the effective conductivity tensor is considerably simplified for isotropic media owing to the dependence of the spectrum Φ only on $k = |\boldsymbol{k}|$. Thus, $\hat{g}$, (44), and $\hat{\psi}$, (47), become functions of k and λ. Since $\hat{\boldsymbol{K}}^{eff}$ depends only on the properties of medium and the latter has no preferential directions in space, the effective conductivity tensor is an isotropic tensor. The general form of the isotropic tensor is given by

$$\hat{K}^{eff}_{ml}(\boldsymbol{k},\lambda) = \left(\delta_{ml}-\frac{k_m k_l}{k^2}\right)\hat{K}^{tr}(k,\lambda)+\frac{k_m k_l}{k^2}\hat{K}^{l}(k,\lambda), \tag{50}$$

where $\hat{K}^{tr}$ and $\hat{K}^{l}$ are the transverse and longitudinal components depending on k and λ only.

The longitudinal and transverse components are determined by calculating the trace and the contraction of (50) and by comparing them with (46) and (49). This yields

$$\hat{K}^{l}(k,\lambda) = K_A[1-\sigma^2\hat{g}(k,\lambda)], \tag{51}$$

$$\hat{K}^{tr}(k,\lambda) = K_A\left[1-\sigma^2\frac{\hat{\psi}(k,\lambda)-\hat{g}(k,\lambda)}{d-1}\right]. \tag{52}$$

The effective conductivity tensor (50), (51) and (52) is given by a full isotropic tensor whose components depend on the two functions $\hat{g}$ and $\hat{\psi}$ of k and λ. For large time $(\lambda\to 0)\,\hat{\psi}\to 1$ and (50), (51) and (52) yield the effective conductivity tensor for steady-state flows

$$\tilde{K}^{l}(k) = K_A[1-\sigma^2\tilde{g}(k)] = K_A\,\tilde{Y}(k), \tag{53}$$

where $\tilde{g}(k) = \hat{g}(k,0)$.

We have shown that the effective conductivity tensor cannot be defined uniquely for a potential vector $\boldsymbol{E}$. Any solenoidal tensor can be added to the kernel $\boldsymbol{K}^{eff}(\boldsymbol{x},t)$

without affecting the distributions of the mean head and the mean velocity. Since the transverse component in the representation (50) is the FT of such a tensor, it does not contribute to the mean head distribution, its gradient or the mean velocity. Thus, the mean head is completely determined by the longitudinal component $\hat{K}^l$-defined by (51). In this sense we can rewrite the averaged Darcy's Law in the Fourier–Laplace domain as follows

$$\langle \hat{\boldsymbol{v}}(\boldsymbol{k},\lambda)\rangle = -\hat{K}^l(k,\lambda)\langle \hat{\boldsymbol{E}}(\boldsymbol{k},\lambda)\rangle; \quad \langle \hat{\boldsymbol{E}}(\boldsymbol{k},\lambda)\rangle = -\mathrm{i}\boldsymbol{k}\langle \hat{h}(\boldsymbol{k},\lambda)\rangle, \tag{54}$$

and regard the longitudinal component $\hat{K}^l(k,\lambda)$ as an effective conductivity of isotropic formations in Fourier–Laplace domain. The averaged Darcy's Law is now simplified to

$$\langle \boldsymbol{v}(\boldsymbol{x},t)\rangle = -\int_0^t \mathrm{d}\tau \int \mathrm{d}\boldsymbol{x}'\, K^l(|\boldsymbol{x}-\boldsymbol{x}'|, t-\tau)\langle \boldsymbol{E}(\boldsymbol{x}',\tau)\rangle, \tag{55}$$

where K^l is the inverse FLT of $\hat{K}^l$.

The averaged Darcy's Law (55) represents a constitutive equation for isotropic media. Coupled with the averaged mass balance equation (23) they constitute a mathematical model of unsteady flows in stationary isotropic media. To solve the system of equations (23), (55) one needs an initial condition (8) and a distribution of the sources ϕ in space and time. The averaged equations can be considered as a model of deterministic flow in an effective medium whose conductivity is a convolution integral operator. Owing to the non-local nature of the constitutive equation, the direction of the mean velocity does not generally coincide with that of the mean head gradient. Only for unidirectional flows (e.g. a flow toward a single well) are the vectors $\langle \boldsymbol{v}\rangle$ and $\langle \boldsymbol{E}\rangle$ collinear (this follows from subsituting the representation $\langle \boldsymbol{E}(\boldsymbol{x},t)\rangle = \mathscr{E}(x,t)\,\boldsymbol{x}/x$ of the mean head gradient into (55)). Note, that in Fourier–Laplace space the mean velocity (54) is parallel to the mean head gradient.

The effective conductivity (51) is a function of two variables, k and λ. We now study its properties by first deriving the bounds restricting the range of variability of $\hat{K}^l(k,\lambda)$. Remember that for uniform steady mean flow the effective conductivity is bounded between harmonic and arithmetic means (2). If the flow is steady-state but non-uniform in the average, the effective conductivity satisfies the narrower inequalities

$$K_H \boldsymbol{I} \leqslant \tilde{\boldsymbol{K}}^{eff}(\boldsymbol{k}) \leqslant \boldsymbol{K}^{eff}, \tag{56}$$

resulting from (43), (44) and (53).

The inequality (56) generalizes (2) for non-uniform mean steady flows. Note that $\tilde{\boldsymbol{K}}^{eff}(\boldsymbol{k})$ is no longer a constant tensor for a given structure of the medium. The heterogeneity structure determines the upper limit $\boldsymbol{K}^{eff}$ in (56) where $\tilde{\boldsymbol{K}}^{eff}(\boldsymbol{k})$ varies in the Fourier domain between the values K_H and $\boldsymbol{K}^{eff}$, approaching lower and upper bounds in the vicinity of and far from the sources of flow non-uniformity. This shows that (56) are best bounds.

For unsteady flows expression (51) shows that the effective conductivity is bounded between the effective conductivity tensor for a steady-state process and the arithmetic mean:

$$\tilde{\boldsymbol{K}}^{eff}(\boldsymbol{k}) \leqslant \hat{\boldsymbol{K}}^{eff}(\boldsymbol{k},\lambda) \leqslant K_A \boldsymbol{I}. \tag{57}$$

Note that unlike (2) and similar to (56) the inequality (57) provides bounds dependent on the structure of the medium. It cannot be improved since both bounds are reached in the flow process. Recalling that the steady-state effective conductivity tensor $\tilde{\boldsymbol{K}}^{eff}$ is bounded between the harmonic mean K_H and the effective conductivity tensor for steady-state uniform flow (56) which in turn is greater than the harmonic mean and less than arithmetic mean (2), the following general bounds are valid for any $\boldsymbol{k}$ and λ:

$$K_H \boldsymbol{I} \leqslant \hat{\boldsymbol{K}}^{eff}(\boldsymbol{k}, \lambda) \leqslant K_A \boldsymbol{I}. \tag{58}$$

The inequality (58) generalizes (2) and (56) for unsteady flows. It depends only on the two values K_H and K_A and in this sense is invariant to the porous medium structure. It is emphasized that for any heterogeneous medium (58) provides the best estimate among others with constant bounds since both K_H and K_A are met in the flow process. Indeed, the low limit is achieved at $\boldsymbol{k} = \infty$ and $\lambda < \infty$, i.e. in the vicinity of the sources of flow non-uniformity, whereas the upper limit occurs at $\lambda \to \infty$, i.e. at the initial stage of the process. It is emphasized that although the estimates (56), (57) and (58) are presented in general form they are derived here at first order in the conductivity variance. Their validity for strongly heterogeneous media has not been proved yet.

Relationship (51) shows that $\hat{K}^l$ is completely determined by the one function $\hat{g}$ of two scalar arguments k and λ and we proceed now with deriving this function for one- two- and three-dimensional flows.

For one-dimensional flow the function $\hat{g}$ results from (44) as follows:

$$\hat{g}(k, \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}p \frac{p^2}{p^2 + \lambda} \Phi(k - p). \tag{59}$$

For $d \geqslant 2$, switching in (44) to the new variable $\boldsymbol{u} = \boldsymbol{p} - \boldsymbol{k}$ and subsequently to the d-dimensional spherical coordinate system yields

$$\hat{g}(k, \lambda) = 1 - \frac{k^d}{2^{d-1}\pi^{(d+1)/2}\Gamma[(d-1)/2]} \int_0^{\infty} \mathrm{d}u\, u^{d-1} F_d(u, \xi)\, \Phi(ku), \tag{60}$$

where $\Gamma(d)$ is the Gamma function, $\xi = \lambda/k^2$ and

$$F_d(u, \xi) = \xi f_{d-2}(u, \xi) + u^2 f_d(u, \xi); \quad f_d(u, \xi) = \int_0^{\pi} \mathrm{d}\varphi \frac{\sin^d \varphi}{1 + \xi + 2u\cos\varphi + u^2}. \tag{61}$$

Calculating f_d from (61) for $d = 0$, 1, 2 and 3, the functions F_2 and F_3 for two- and three-dimensional flows are obtained as follows:

$$F_2(u, \xi) = \frac{\pi}{4}\left[1 + u^2 + \xi - \frac{(1 - u^2 - \xi)^2}{[(1 + u^2 + \xi)^2 - 4u^2]^{1/2}}\right], \tag{62}$$

$$F_3(u, \xi) = \frac{\pi}{2}\left[1 + u^2 + \xi - \frac{(1 - u^2 - \xi)^2}{4u} \ln \frac{(1 + u)^2 + \xi}{(1 - u)^2 + \xi}\right]. \tag{63}$$

Expressions (59)–(63) show explicitly the dependence of $\hat{g}$ and $\hat{\boldsymbol{K}}^{eff}$ on the shape of the correlations. In particular, for steady-state flows the function $\tilde{g}(k) = \hat{g}(k, 0)$ is simplified to

$$\hat{g}(k) = 1 - \frac{k^d}{2^{d-1}\pi^{(d+1)/2}\Gamma[(d-1)/2]} \int_0^{\infty} \mathrm{d}u\, u^{d+1} f_d(u, 0)\, \Phi(ku), \tag{64}$$

with

$$f_2(u, 0) = \frac{\pi}{1 + u^2 + |1 - u^2|}; \quad f_3(u, 0) = \frac{1 + u^2}{2u^2} - \frac{(1 - u^2)^2}{4u^3} \ln\left(\frac{1 + v}{1 - v}\right)^2. \tag{65}$$

Several examples of closed expressions of $\tilde{g}(k)$ can be found in Indelman & Abramovich (1994*b*).

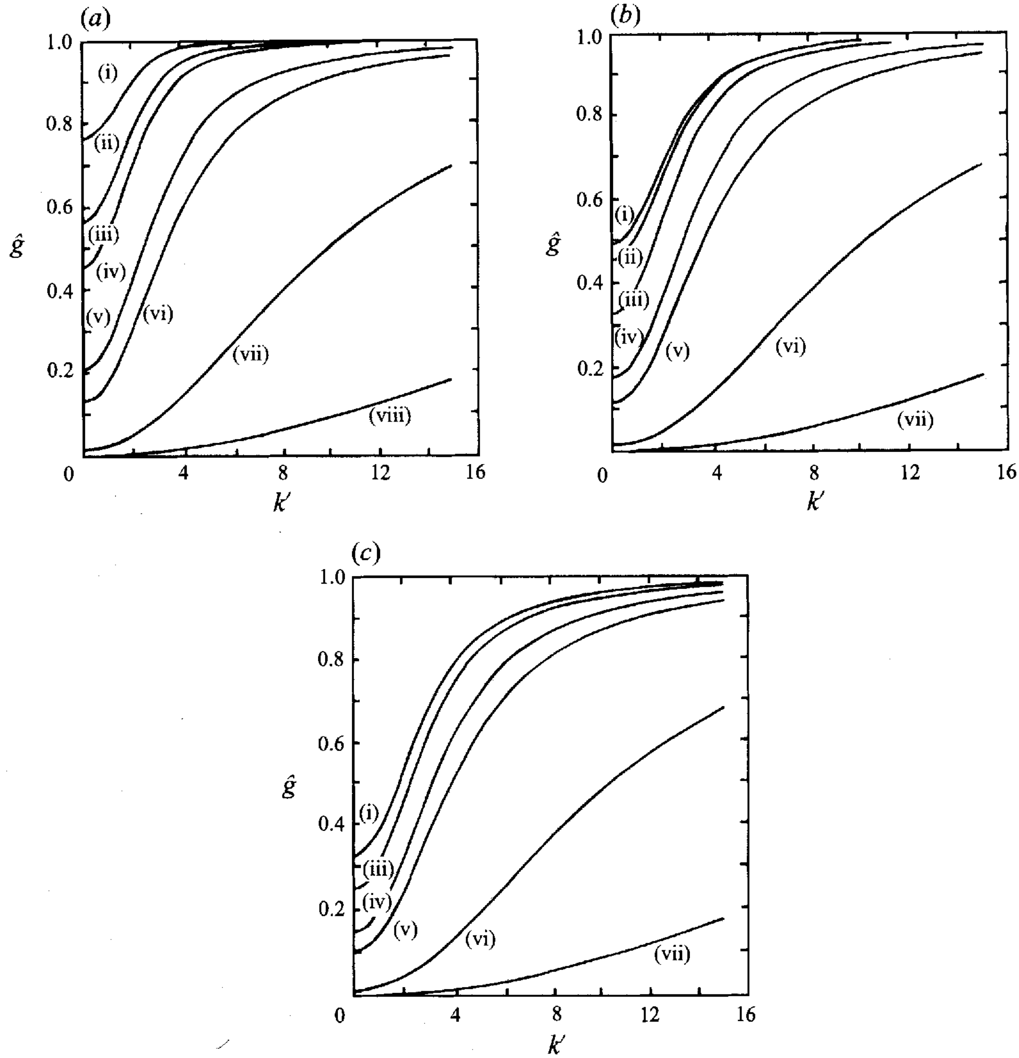

FIGURE 1. Dependence of $\hat{g}$ on $k' = kl$ and $\lambda' = \lambda l^2$ and for a Gaussian conductivity correlation. (*a*) One-dimensional flow: (i) $\lambda' = 0$, (ii) $\lambda' = 0.1$, (iii) $\lambda' = 0.5$, (iv) $\lambda' = 1$, (v) $\lambda' = 5$, (vi) $\lambda' = 10$, (vii) $\lambda' = 10^2$, (viii) $\lambda' = 10^3$. (*b*) Two-dimensional flow and (*c*) three-dimensional flow: (i) $\lambda' = 0$, (ii) $\lambda' = 0.1$, (iii) $\lambda' = 1$, (iv) $\lambda' = 5$, (v) $\lambda' = 10$, (vi) $\lambda' = 10^2$, (vii) $\lambda' = 10^3$.

The function $\hat{g}(k, \lambda)$ was calculated for a Gaussian correlation as a function of $k' = kl$ and for different values of $\lambda' = \lambda l^2$ where l is the scale of heterogeneity. It is depicted in figures 1(*a*), 1(*b*) and 1(*c*) for one-, two- and three-dimensional flows, respectively. The effective conductivity $\hat{K}^l = K_A[1-\sigma^2\hat{g}]$ depends on the variance of ϵ. For given σ^2 it is seen from figure 1 how $\hat{K}^l$ varies in the Fourier–Laplace space for the entire process of unsteady average flow. In agreement with the general results for any k the longitudinal component $\hat{K}^l(k, \lambda)$ varies from the effective value for steady flow $\hat{K}^l(k, 0)$ at small λ (large times) to the arithmetic mean K_A at large λ (small times). It is seen that at small k, i.e. at distances from the sources of flow non-uniformity much larger than the integral scale, the effective property varies from the effective value $K_A(1-\sigma^2/d)$ for uniform steady flow to the arithmetic mean. This generalizes the Dagan's (1982) result obtained for unsteady flows without sources. It is seen that close to the sources of flow non-uniformity (e.g. in the vicinity of a well or of steep gradients in initial head distribution) the effective conductivity varies between the harmonic and arithmetic means approaching the bounds at small and large λ (at large and small times, respectively). At large times (small λ) $\hat{K}^l$ varies from the harmonic mean at wells to the uniform steady-state effective value far from the well in agreement with Shvidler's (1966) asymptotic limits of the mean Green function for a flow toward a single source.

8. Asymptotic analysis of the averaged Darcy's Law for isotropic media

Although some limit values of the effective conductivity tensor were obtained in the previous Sections it is important to investigate the asymptotic dependence of average flows on distances from the sources of mean flow non-uniformity much greater than the scale of heterogeneity l (small-scale heterogeneity (Shvidler 1985)). To derive this asymptotic behaviour we consider the effective Darcy's Law in the Fourier–Laplace space (54) with $\hat{K}^l$ given by (51) and (44). For $k' = kl \ll 1$, the spectrum $\Phi(\boldsymbol{k}-\boldsymbol{p})$ can be expanded in a power series of $\boldsymbol{k}$ as follows:

$$\Phi(\boldsymbol{k}-\boldsymbol{p}) \simeq \Phi(p)-(\boldsymbol{k}\cdot\nabla)\,\Phi(p)+\tfrac{1}{2}(\boldsymbol{k}\cdot\nabla)^2\,\Phi(p). \tag{66}$$

Expansion (66) is substituted in (44). Since the spectrum is an even function, the second term in (66) does not contribute to $\hat{g}(k,\lambda)$. It is convenient to introduce a new variable $\boldsymbol{u} = l\boldsymbol{p}$ and a function $\Psi(u) = l^{-d}\Phi(p)$. The function $\hat{g}$, (44), becomes after some algebraic operations

$$\hat{g}(k,\lambda') \simeq \check{\beta}_d(\lambda')+\check{\alpha}_d(\lambda')\,l^2k^2 \quad (kl \ll 1), \tag{67}$$

where $\lambda' = \lambda l^2$ and

$$\check{\beta}_d(\lambda') = \int \frac{\mathrm{d}\boldsymbol{u}}{(2\pi)^d}\frac{(\boldsymbol{k}\cdot\boldsymbol{u})^2}{k^2(u^2+\lambda')}\,\Psi(u) = \frac{1}{d}\int\frac{\mathrm{d}\boldsymbol{u}}{(2\pi)^d}\frac{u^2}{u^2+\lambda'}\,\Psi(u), \tag{68}$$

$$\check{\alpha}_d(\lambda') = \frac{1}{2}\int\frac{\mathrm{d}\boldsymbol{u}}{(2\pi)^d}\frac{(\boldsymbol{k}\cdot\boldsymbol{u})^2}{k^2(u^2+\lambda')}\left[\frac{(\boldsymbol{k}\cdot\boldsymbol{u})^2}{k^2u}\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{u}\frac{\mathrm{d}\Psi}{\mathrm{d}u}\right)+\frac{1}{u}\frac{\mathrm{d}\Psi}{\mathrm{d}u}\right]$$

$$= \frac{1}{2d}\int\frac{\mathrm{d}\boldsymbol{u}}{(2\pi)^d}\frac{u^2}{u^2+\lambda'}\left[\frac{3u}{d+2}\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{u}\frac{\mathrm{d}\Psi}{\mathrm{d}u}\right)+\frac{1}{u}\frac{\mathrm{d}\Psi}{\mathrm{d}u}\right]. \tag{69}$$

In (68) and (69) we switched to spherical variables and integrated over the angle.

The asymptotics (67) shows that far from the sources of the mean flow non-uniformity the averaged Darcy's Law is expressed in the form

$$\langle \boldsymbol{v}(\boldsymbol{x},t')\rangle = -K_A\left\{\langle \boldsymbol{E}(\boldsymbol{x},t')\rangle-\sigma^2\int_0^{t'}\mathrm{d}\tau[\beta_d(t'-\tau)+\alpha_d(t'-\tau)\,\nabla^2]\,\langle \boldsymbol{E}(\boldsymbol{x},\tau)\rangle\right\} \quad (kl \ll 1), \tag{70}$$

where $t' = t/l^2$ and $\beta_d(t')$ and $\alpha_d(t')$ are the inverse LT of $\check{\beta}_d(\lambda')$ and $\check{\alpha}_d(\lambda')$, respectively. It is easy to check that $\beta_d(t') = -\mathrm{d}b_d(t')/\mathrm{d}t'$ where $b_d(t')$ is a function obtained by Dagan (1982) and is given by expression (A 2) of his article. This is expected, since at distances far from the sources of the non-uniformity ($\boldsymbol{k} = 0$) the asymptotic (67) should lead to (39), (40) if $\langle \boldsymbol{E}(x,t)\rangle$ varies slowly in time. For exponential covariance the function $\beta_d(t)$ is easily determined from expressions (A 3)–(A 5) of Dagan's (1982) paper for $d = 1$, 2 and 3. The corresponding expressions for $\beta_d(t')$ for a Gaussian covariance are obtained as

$$\beta_d(t') = 2(1+4t')^{-(1+d/2)}. \tag{71}$$

The second term of the integrand in (70) describes the effect of the flow non-uniformity on the velocity field at distances far from the sources. The kernel $\alpha_d(t')$ results from (69) after switching to spherical coordinates and applying the inverse LT:

$$\alpha_d(t') = \frac{1}{2^d \pi^{d/2} \Gamma(d/2)\, d} \int_0^\infty \mathrm{d}u\, u^d\, \mathrm{e}^{-u^2 t'} \left[\frac{3u^2}{d+2} \frac{\mathrm{d}}{\mathrm{d}u} \left(\frac{1}{u} \frac{\mathrm{d}\Psi(u)}{\mathrm{d}u} \right) + \frac{\mathrm{d}\Psi(u)}{\mathrm{d}u} \right]. \tag{72}$$

In particular, for a Gaussian covariance (72) yields

$$\alpha_d(t') = \frac{1-2t'}{(1+4t')^{2+d/2}}. \tag{73}$$

It is seen that for small times and at large distances from the sources of non-uniformity the averaged Darcy's Law is approximated by a local relation with arithmetic mean standing for the effective conductivity: $\langle \boldsymbol{v} \rangle = -K_A \langle \boldsymbol{E} \rangle$. In contrast, for large times the asymptotics (70) leads to the steady-state one derived previously by Indelman & Abramovich (1994*b*). Thus, for a Gaussian correlation we have using (71) and (73)

$$\begin{aligned} \langle \boldsymbol{v}^{st}(\boldsymbol{x}) \rangle &= \lim_{t' \to \infty} \langle \boldsymbol{v}(\boldsymbol{x}, t') \rangle = -\lim_{\lambda' \to 0} \lambda' [\breve{\beta}_d(\lambda') + l^2 \breve{\alpha}_d(\lambda') \nabla^2] \langle \breve{\boldsymbol{E}}(\boldsymbol{x}, \lambda') \rangle \\ &= -K_A \left\{ 1 - \sigma^2 \int_0^\infty \mathrm{d}\tau [\beta_d(\tau) + l^2 \alpha_d(\tau) \nabla^2] \right\} \langle \boldsymbol{E}^{st}(\boldsymbol{x}) \rangle \\ &= -K^{eff} \left[1 - \sigma^2 \frac{d-1}{2d(d+2)} l^2 \nabla^2 \right] \langle \boldsymbol{E}^{st}(\boldsymbol{x}) \rangle, \end{aligned} \tag{74}$$

where $K^{eff} = K_A(1 - \sigma^2/d)$ is the effective conductivity for steady-state uniform flow. Similar asymptotic expressions can be derived for any covariance of finite integral scale.

The important question is what are conditions needed for neglecting the spatial non-local term in (70). Let l_E be the characteristic scale of the mean head gradient defined by $\nabla^2 \langle \boldsymbol{E}(\boldsymbol{x}, t') \rangle \sim l_E^{-2} \langle \boldsymbol{E}(\boldsymbol{x}, t') \rangle$. It follows from (70) that the term with ∇^2 can be neglected if the characteristic scale of the flow non-uniformity l_E satisfies the condition

$$l_E \gg \sigma l \gamma_d(t'); \quad \gamma_d(t') = \left(\int_0^{t'} \mathrm{d}\tau\, \alpha_d(\tau) \right)^{1/2}. \tag{75}$$

Thus inside the flow subdomains, where the characteristic scale l_E of the mean head gradient obeys (75), the averaged Darcy's Law can be approximated by a relation local in space and non-local in time:

$$\langle \boldsymbol{v}(\boldsymbol{x}, t') \rangle = -K_A \left\{ \langle \boldsymbol{E}(\boldsymbol{x}, t') \rangle - \sigma^2 \int_0^{t'} \mathrm{d}\tau\, \beta_d(t' - \tau) \langle \boldsymbol{E}(\boldsymbol{x}, \tau) \rangle \right\}. \tag{76}$$

Hence, for a Gaussian correlation relation (76) is applicable if

$$l_E \gg \sigma l \gamma_d(t'); \quad \gamma_d(t') = \frac{1}{[2d(d+2)]^{1/2}} \left[d - 1 - \frac{d-1-2(d+2)\,t'}{(1+4t')^{1+d/2}} \right]^{1/2}. \tag{77}$$

It is seen that condition (77) depends on the stage of the process. It follows from (73) that $\gamma_d(t')$ has a maximum at $t' = 1/2$ for any d and $\gamma_d(1/2)$ varies slightly in d. Therefore the sufficient condition of applicability of (76) for $d = 1, 2, 3$ is given by

$l_E \gg 0.31\sigma l$. For comparison, the corresponding condition for l_E in steady flow results from (74) as follows:

$$l_E^{st} \gg \sigma l \left[\frac{d-1}{2d(d+2)}\right]^{1/2},$$

or $l_E^{st} \gg 0.26\sigma l$ for $d = 2$ and 3.

9. Summary

The aim of the present study is to develop a procedure of averaging unsteady flows in heterogeneous formations of stationary conductivity and to study the properties of the mean flows. This procedure generalizes the approach proposed previously for steady flows (Indelman & Abramovich 1994*b*) and is based on applying the Fourier–Laplace transform to the flow equations which are further solved by perturbation methods. The advantage of this approach is that in Fourier–Laplace domain the dependences of the mean flow quantities (mean head, its gradient and mean velocity) on the heterogeneity structure on one hand and on the initial head and the distribution of the sources on the other hand are decoupled and have a local structure. Applying the inverse FLT permits one to determine the mean flow in the physical domain.

One of the central problems of flows in media of random structure is in deriving the averaged Darcy's Law. The main result of our study is a linear local relation between mean velocity and mean head gradient in Fourier–Laplace variables. The coefficient of proportionality which we call the effective conductivity tensor depends only on the statistical properties of a random field $K(\boldsymbol{x})$. The inverse FLT yields the averaged Darcy's Law in physical variables. This relation defines the effective conductivity as a convolution integral operator over space and time operating on the mean head gradient. The kernel of this integral transform is the inverse FLT of the effective conductivity tensor and is completely known for a given structure of the porous medium.

Our second result is related to the mean head distribution. It is shown that the FLT of the mean head is given by a product of two functions. The first function is a FLT of the solution of the problem for homogeneous media. The second one depends only on the structure of heterogeneities and is simply related to the effective conductivity tensor. Thus the effect of the heterogeneities on the mean head is concentrated into one function $\hat{Y}$ which is of the utmost importance in describing average flows. In particular, this function determines the FLT of the fundamental solution of the mean flow equations which is referred to as the mean Green function.

We study further the general properties of the averaged flows. It is shown that the kernel of the effective conductivity is not defined uniquely. However, this non-uniqueness does not affect the distributions of the mean head, its gradient or the mean velocity. We show that the effective conductivity for unsteady flows generalizes the concept of the effective property for steady-state non-uniform flows. The latter, in turn, is a generalization of the uniform version of effective properties. These properties are manifested in the following relationships between the effective conductivities of different processes: $\boldsymbol{K}^{eff} = \tilde{\boldsymbol{K}}^{eff}(\boldsymbol{k}=0) = \hat{\boldsymbol{K}}^{eff}(\boldsymbol{k}=0, \lambda=0)$ and $\tilde{\boldsymbol{K}}^{eff}(\boldsymbol{k}) = \hat{\boldsymbol{K}}^{eff}(\boldsymbol{k}, \lambda=0)$. It is emphasized that in general the averaged Darcy's Law has a non-local structure with effective conductivity being a functional of the conductivity statistics.

These general results are applied to flows in weakly heterogeneous media to derive the first-order approximation of the effective conductivity tensor in terms of the heterogeneity spectrum. We determine a general structure of the effective conductivity tensor of isotropic media by deriving expressions for the longitudinal and transverse components of the isotropic tensor. We note that the transverse component of the effective conductivity tensor does not contribute to the mean head distribution and the mean velocity field and in this sense can be dropped. The averaged Darcy's Law is now expressed by the convolution integral operator whose kernel is a scalar function equal to the longitudinal component. The results are applicable to any number of space dimensions. The FLT of the longitudinal component $\hat{K}^l$ is further investigated for one- two- and three-dimensional flows. We develop an asymptotic model of the effective Darcy's Law applicable for large distances from the sources of mean flow non-uniformity (small-scale heterogeneity). The constitutive equation can be localized in space for mean flows whose characteristic scale is much greater than the correlation scale.

The new bounds for the effective conductivity tensor generalizing those known for steady uniform flows are proved for weakly heterogeneous media. It is recalled that the effective conductivity of steady uniform flows is bounded by harmonic and arithmetic means. This is an inequality which bounds the possible values of the effective property for different structures of the porous media. Our new assessments have a dynamic nature. The first inequality bounds the effective conductivity tensor for unsteady flow between the effective conductivity tensor for steady non-uniform flow and the arithmetic mean. In turn, the effective conductivity tensor of steady non-uniform flows is bounded between the harmonic mean and the effective conductivity for uniform mean flows. We note that these estimates cannot be improved since both lower and upper bounds are met in the flow process. The weaker assessment of the effective conductivity tensor coincides with the aforementioned one for steady uniform cases.

It is emphasized that the first-order results of our study are applicable for small coefficients of variations of the conductivity. This is quite a restrictive assumption which is usually not met in natural formations. It is well known that expanding the head in series in the log-conductivity fluctuations considerably alleviates this restriction. The perturbation series applied in the paper was chosen to shorten the derivations and reformulation of the final results in terms of the log-conductivity statistics is straightforward.

Summarizing, the mathematical model of unsteady flows in media of stationary conductivity is developed to be applicable to any initial condition and distribution of the sources. The constitutive equation of the effective medium, the average Darcy's Law, defines the effective conductivity as a convolution integral operator in space and time. The properties of the effective conductivity tensor are studied for general structures of heterogeneity and for small fluctuations of the conductivity.

Appendix A. Averaging of the flow equations in Fourier–Laplace space

The equations of the average flow are derived by perturbation methods using the procedure developed recently for steady-state flows. (Indelman & Abramovich 1994*b*). Consider an approximation of the head of the order of n given by (11). Replacing $\hat{h}^{(n-1)}$ by (11) for $n-1$ and continuing this procedure subsequently yields the following expression of the FLT of $h^{(n)}$:

$$\hat{h}^{(n)}(\boldsymbol{k},\lambda) = \int \cdots \int \frac{\mathrm{d}\boldsymbol{p}_1 \ldots \mathrm{d}\boldsymbol{p}_n}{(2\pi)^{nd}} \frac{\boldsymbol{k}\cdot\boldsymbol{p}_1}{k^2+\lambda} \frac{\boldsymbol{p}_1\cdot\boldsymbol{p}_2}{p_1^2+\lambda} \cdots \frac{\boldsymbol{p}_{n-1}\cdot\boldsymbol{p}_n}{p_{n-1}^2+\lambda}$$
$$\times\, \tilde{\varepsilon}(\boldsymbol{k}-\boldsymbol{p}_1)\,\tilde{\varepsilon}(\boldsymbol{p}_1-\boldsymbol{p}_2)\ldots\tilde{\varepsilon}(\boldsymbol{p}_{n-1}-\boldsymbol{p}_n)\,\hat{h}^{(0)}(\boldsymbol{p}_n,\lambda) \quad (n = 1,2,\ldots), \quad \text{(A 1)}$$

where the zero-order approximation of the head is given by its FLT (12).

The mean of the n-order approximation of the head is obtained by averaging (A 1). Since the conductivity is a stationary space function, the correlation in (A 1) can be expressed in terms of the spectrum function by

$$\langle \tilde{\epsilon}(\boldsymbol{k}-\boldsymbol{p}_1)\,\tilde{\epsilon}(\boldsymbol{p}_1-\boldsymbol{p}_2)\ldots\tilde{\epsilon}(\boldsymbol{p}_{n-1}-\boldsymbol{p}_n)\rangle = (2\pi)^d\,\Phi(\boldsymbol{k},\boldsymbol{p}_1,\ldots,\boldsymbol{p}_{n-1})\,\delta(\boldsymbol{k}-\boldsymbol{p}_n), \quad \text{(A 2)}$$

where $\Phi(\boldsymbol{k},\boldsymbol{p}_1,\ldots,\boldsymbol{p}_{n-1})$ is the Fourier transform of the n-point correlation function of the process $\epsilon(\boldsymbol{x})$ defined by (14). Averaging (A 1) and substituting (A 2) yields

$$\langle \hat{h}^{(n)}(\boldsymbol{k},\lambda)\rangle = \frac{\boldsymbol{k}\cdot\boldsymbol{\Omega}^{(n)}(\boldsymbol{k},\lambda)\cdot\boldsymbol{k}}{k^2+\lambda}\,\hat{h}^{(0)}(\boldsymbol{k},\lambda) \quad (n = 2,3,\ldots), \quad \text{(A 3)}$$

where $\boldsymbol{\Omega}^{(n)}(\boldsymbol{k},\lambda)$ is a tensor given by

$$\Omega^{(n+1)}_{ml} = \int\cdots\int\frac{\mathrm{d}\boldsymbol{p}_1\ldots\mathrm{d}\boldsymbol{p}_n}{(2\pi)^{nd}}\frac{p_{1,m}\,p_{n,l}}{p_n^2+\lambda}\frac{\boldsymbol{p}_1\cdot\boldsymbol{p}_2}{p_1^2+\lambda}\cdots\frac{\boldsymbol{p}_{n-1}\cdot\boldsymbol{p}_n}{p_{n-1}^2+\lambda}\,\Phi(\boldsymbol{k},\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n) \quad (n = 1,2,\ldots). \quad \text{(A 4)}$$

The FLT of the mean head now results from (A 3):

$$\langle \hat{h}(\boldsymbol{k},\lambda)\rangle = \left[1+\frac{\boldsymbol{k}\cdot\boldsymbol{\Omega}(\boldsymbol{k},\lambda)\cdot\boldsymbol{k}}{k^2+\lambda}\right]\hat{h}^{(0)}(\boldsymbol{k},\lambda), \quad \text{(A 5)}$$

where $\hat{h}^{(0)}(\boldsymbol{k},\lambda)$ is completely defined by the initial head distribution and the source function and $\boldsymbol{\Omega}$ is a tensor given by the series

$$\boldsymbol{\Omega}(\boldsymbol{k},\lambda) = \sum_{n=2}^{n=\infty}\boldsymbol{\Omega}^{(n)}(\boldsymbol{k},\lambda); \quad \boldsymbol{\Omega}^{(n)} = O(\sigma^n). \quad \text{(A 6)}$$

In general the mean head distribution depends on the properties of the medium, the initial condition and the distribution of sources. The remarkable property of solution (A 5) is that the FLT of the mean head is obtained as a product of two functions. The first term on the right-hand side of (A 5) depends only on the structure of medium, whereas the second one is completely determined by the initial head distribution and the source function. It is precisely this splitting of the solution into two terms, one of which carries information about the structure of heterogeneity whereas the second term specifies the initial and source conditions, that permits one to derive the local averaged Darcy's Law in the Fourier–Laplace domain.

The FLT of the mean head gradient now results from (4) as

$$\langle \hat{\boldsymbol{E}}(\boldsymbol{k},\lambda)\rangle = -\mathrm{i}\boldsymbol{k}\langle \hat{h}(\boldsymbol{k},\lambda)\rangle.$$

The FLT of the mean velocity is obtained from (4) as follows:

$$\langle \hat{\boldsymbol{v}}(\boldsymbol{k},\lambda)\rangle = -K_A\left[\langle \hat{\boldsymbol{E}}(\boldsymbol{k},\lambda)\rangle - \int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^d}\langle \tilde{\epsilon}(\boldsymbol{k}-\boldsymbol{p})\,\hat{\boldsymbol{E}}(\boldsymbol{p},\lambda)\rangle\right]. \quad \text{(A 7)}$$

Using (A 1) the integral in (A 7) can be rewritten in the form

$$\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^d}\langle \tilde{\epsilon}(\boldsymbol{k}-\boldsymbol{p})\,\hat{\boldsymbol{E}}(\boldsymbol{p},\lambda)\rangle = -\mathrm{i}\sum_{n=1}^{\infty}\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^d}\boldsymbol{p}\langle \tilde{\epsilon}(\boldsymbol{k}-\boldsymbol{p})\,\hat{h}^{(n)}(\boldsymbol{p},\lambda)\rangle$$

$$= -i\sum_{n=1}^{\infty}\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^d}\boldsymbol{p}\int\cdots\int\frac{\mathrm{d}\boldsymbol{p}_1\ldots\mathrm{d}\boldsymbol{p}_n}{(2\pi)^{nd}}\frac{\boldsymbol{p}\cdot\boldsymbol{p}_1}{p^2+\lambda}\frac{\boldsymbol{p}_1\cdot\boldsymbol{p}_2}{p_1^2+\lambda}\cdots\frac{\boldsymbol{p}_{n-1}\cdot\boldsymbol{p}_n}{p_{n-1}^2+\lambda}$$

$$\times\langle \tilde{\epsilon}(\boldsymbol{k}-\boldsymbol{p})\,\tilde{\epsilon}(\boldsymbol{p}-\boldsymbol{p}_1)\,\tilde{\epsilon}(\boldsymbol{p}_1-\boldsymbol{p}_2)\ldots\tilde{\epsilon}(\boldsymbol{p}_{n-1}-\boldsymbol{p}_n)\rangle\,\hat{h}^{(0)}(\boldsymbol{p}_n,\lambda).$$

Substituting (A 2) into this expression transforms the latter to

$$-\mathrm{i}\hat{h}^{(0)}(\boldsymbol{k},\lambda)\sum_{n=1}^{\infty}\int\frac{\mathrm{d}\boldsymbol{p}_1\ldots\boldsymbol{p}_n}{(2\pi)^{nd}}\frac{\boldsymbol{p}_1\cdot\boldsymbol{p}_2}{p_1^2+\lambda}\cdots\frac{\boldsymbol{p}_{n-1}\cdot\boldsymbol{p}_n}{p_{n-1}^2+\lambda}\frac{\boldsymbol{p}_n\cdot\boldsymbol{k}_n}{p_n^2+\lambda}\boldsymbol{p}_1\,\Phi(\boldsymbol{k},\boldsymbol{p}_1,\ldots,\boldsymbol{p}_m).$$

Using (A 5) and (A 4) we have

$$\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^d}\langle\tilde{\epsilon}(\boldsymbol{k}-\boldsymbol{p})\,\hat{E}_m(\boldsymbol{p},\lambda)\rangle=-\mathrm{i}\hat{h}^{(0)}(\boldsymbol{k},\lambda)\sum_{n=1}^{\infty}\Omega_{ml}^{(n+1)}(\boldsymbol{k},\lambda)\,k_l$$
$$=\hat{Y}(\boldsymbol{k},\lambda)\,\Omega_{ml}(\boldsymbol{k},\lambda)\,\langle\hat{E}_l(\boldsymbol{k},\lambda)\rangle,\quad\text{(A 8)}$$

where
$$\hat{Y}(\boldsymbol{k},\lambda)=\left[1+\frac{\boldsymbol{k}\cdot\boldsymbol{\Omega}(\boldsymbol{k},\lambda)\cdot\boldsymbol{k}}{k^2+\lambda}\right]^{-1}.$$

Substituting (A 8) into (A 7) yields

$$\langle\hat{v}_m(\boldsymbol{k},\lambda)\rangle=-K_A[\delta_{ml}-\hat{Y}(\boldsymbol{k},\lambda)\,\Omega_{ml}(\boldsymbol{k},\lambda)]\,\langle\hat{E}_l(\boldsymbol{k},\lambda)\rangle.\quad\text{(A 9)}$$

It follows from (A 4) and (A 6) that $\boldsymbol{\Omega}$ and, concurrently, $\hat{Y}$ depend only on the statistics of the conductivity K and do not depend on the initial head distribution or source function.

Appendix B. Effective conductivity for one-dimensional flow

For one-dimensional flow

$$\hat{Y}(k,\lambda)=\left[1+\frac{k^2}{k^2+\lambda}\Omega(k,\lambda)\right]^{-1},$$

and (19) becomes
$$\hat{K}^{eff}(k,\lambda)=\frac{1-(\lambda/\lambda+k^2)\,\Omega(k,\lambda)}{1+(k^2/\lambda+k^2)\,\Omega(k,\lambda)}K_A.\quad\text{(B 1)}$$

Here $\Omega=\Sigma_{n=2}^{\infty}\Omega^{(n)}$ with $\Omega^{(n)}$, (15) simplifying to

$$\Omega^{(n+1)}(k,\lambda)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{\mathrm{d}p_1\ldots\mathrm{d}p_n}{(2\pi)^n}\prod_{l=1}^{n}\frac{p_l^2}{\lambda+p_l^2}\Phi(k,p_1,\ldots,p_n)\quad(n=1,2,\ldots).\quad\text{(B 2)}$$

Substituting the definition (14) of the spectrum function $\Phi(k,p_1,\ldots,p_n)$ into (B 2) yields

$$\Omega^{(n+1)}(k,\lambda)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{d}x_1\ldots dx_n\exp(\mathrm{i}kx_1)\,\langle\epsilon(0)\,\epsilon(x_1)\ldots\epsilon(x_n)\rangle$$
$$\times\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{\mathrm{d}p_1\ldots\mathrm{d}p_n}{(2\pi)^n}\prod_{l=1}^{n}\frac{p_l^2}{\lambda+p_l^2}\exp[\mathrm{i}p_1(x_2-x_1)$$
$$+\ldots+\mathrm{i}p_{n-1}(x_n-x_{n-1})-\mathrm{i}p_n x_n].\quad\text{(B 3)}$$

Integrating over $p_1,\ldots,p_n$ in (B 3) leads to

$$\Omega^{(n+1)}(k,\lambda)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{d}x_1\ldots dx_n\exp(\mathrm{i}kx_1)\,\rho(x_1,\ldots,x_n)$$
$$\times\prod_{l=1}^{n}\left[\delta(a_l)-\frac{\lambda^{1/2}}{2}\exp(-|a_l|\,\lambda^{1/2}\right],\quad\text{(B 4)}$$

where $$a_l = x_{l+1} - x_l, \quad l = 1, \ldots, n-1, \quad a_n = x_n. \tag{B 5}$$

Expression (B 4) shows explicitly that $\Omega^{(n)}$ and hence Ω and $\hat{K}^{eff}(k, \lambda)$ depend on the correlation structure of the conductivity field, i.e. $\hat{K}^{eff}(k, \lambda)$ is a functional of the conductivity spatial moments. However, for large times (small λ) this dependence vanishes and $\hat{K}^{eff}(k, \lambda)$ tends to the harmonic mean which is the effective conductivity of the steady-state one-dimensional flow. Indeed, for $\lambda = 0$ (B 4) becomes

$$\Omega^{(n+1)}(k) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \mathrm{d}x_1 \ldots \mathrm{d}x_n \exp(\mathrm{i}kx_1)\, \rho(x_1), \ldots, x_n) \times \delta(x_2 - x_1) \ldots \delta(x_n - x_{n-1})\, \delta(x_n) = \langle \epsilon^{n+1}(0) \rangle, \tag{B 6}$$

and therefore $$\Omega = \sum_{n=2}^{\infty} \langle \epsilon^n(0) \rangle = \left\langle \frac{1}{1-\epsilon} \right\rangle - 1. \tag{B 7}$$

Substituting (B 7) into (B 1) for $\lambda = 0$ and recalling that $K(x) = K_A[1 - \epsilon(x)]$ yields finally

$$\hat{K}^{eff} = \frac{K_A}{\langle (1-\epsilon)^{-1} \rangle} = \frac{1}{\langle K^{-1} \rangle} = K_H. \tag{B 8}$$

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